Linear Equation Theory - 2
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Statement of Prerequisite Skills
Complete all previous TLM modules before beginning this module.

Required Supporting Materials
Internet Explorer 5.5 or greater.
Macromedia Flash Player.

Rationale
Why is it important for you to learn this material?
This module expands on the knowledge gained in the linear equation theory module. The skills gained in this module will allow you to solve problems involving straight lines when only partial information about the problem is given.

Learning Outcome
When you complete this module you will be able to…
Solve problems involving straight lines.

Learning Objectives
1. Change a given linear equation into an equivalent form.
2. Determine the equation of a line using the point–slope form of a line, given two points on the line or given one point and the slope of the line.
3. Determine the slope of an oblique line given its equation.
4. Determine the equation of a line given its graph.
5. Determine the slope, x–intercept, y–intercept, and/or the equation of a horizontal or vertical line.
6. Apply the relationships that occur between lines that are parallel and those that occur between lines that are perpendicular.
7. Solve word problems relating to straight lines.

Connection Activity
Consider a triangle with vertices $A(-3, 4)$, $B(4, 1)$, $C(2, -2)$. What would you do to determine the length from point $A$ to Point $C$?
OBJECTIVE ONE

When you complete this objective you will be able to…

Change a given linear equation into an equivalent form.

Exploration Activity

If calculators or computers are to be used in solving many questions in coordinate geometry, it is often necessary to place expressions in a standardized format. For example the line \( y = -x + 2 \) and the line \( x + y = 2 \) represent the same relationship between \( x \) and \( y \). This objective reviews the manipulation of algebraic expressions into standardized form.

1. **Standard form**
   
   \[ Ax + By = C \]
   
   A, B, C are integers and \( A \geq 0 \)

2. **y intercept form (also called slope/intercept form)**
   
   \[ y = mx + b \]
   
   \( m \) and \( b \) are any real numbers where \( m \) is the slope of the line and \( b \) represents the \( y \) – intercept

3. **Point-slope form**

   \[ y - y_1 = m(x - x_1) \]

   \( x_1, y_1, \) and \( m \) are any real numbers where \( m \) is the slope of the line and \( (x_1, y_1) \) represents a point on the line

**Comment:**

1. The student should be aware that the slope/intercept form (form 2 above) is very useful as a “quick” way of determining the slope of the line. This is possible because the coefficient of \( x \) in this form is the slope. The student must be careful though, to make sure the coefficient of \( y \) is +1 when using this form of the equation of a straight line.

2. Form 1 (Standard) is the form we most often start or finish with.

3. Form 3 (point-slope) will be dealt with in detail in Objective 2. It is a very useful form.
EXAMPLE 1

Express \( y - 5 = 3(x - 4) \) in standard form and state the values of A, B, and C.

SOLUTION:
\[
y - 5 = 3(x - 4)
\]
Remove braces to get:
\[
y - 5 = 3x - 12
\]
Rearrange terms:
\[
-3x + y = -7
\]
Multiply through by -1 to get:
\[
3x - y = 7
\]

Compare \( 3x - y = 7 \) with \( Ax + By = C \)

then we see that: \( A = 3, B = -1 \) and \( C = 7 \)

Note: You must rearrange the equation so that A is positive!!!

EXAMPLE 2

Express \( 3x - y - 7 = 0 \) in \( y \) intercept form and state the values of \( m \) and \( b \).

SOLUTION:
\[
3x - y - 7 = 0
\]
\[
-y = -3x + 7
\]
\[
y = 3x - 7
\]

Compare \( y = 3x - 7 \) with \( y = mx + b \)
and see that: \( m = 3 \) and \( b = -7 \)
Experiential Activity One

1. Express \(-4x - y + 7 = 0\) in the form \(y = mx + b\).

2. Express \(6x + 2y = 7\) in the form \(y = mx + b\).

3. Express \(y = -5x - 3\) in the form \(Ax + By = C\).

4. Express \(y = \frac{2x - 1}{3}\) in the form \(Ax + By = C\).

5. Express \(y - 3 = \frac{2}{3}(x + 4)\) in the form \(Ax + By = C\). Show Me.

6. Express \(2x + 4y = 1\) in \(y = mx + b\) form.

7. Express \(y = 2x - 5\) in the form \(Ax + By = C\).

8. Express \(y = \frac{2}{3}x + 5\) in the form \(Ax + By = C\) and state values for \(A, B, C\).

9. Express \(7x - 4y + 2 = 0\) in the form \(y = mx + b\) and state values for \(m, b\).

10. Express \(y = \frac{5}{4}x + 2\) in the form \(Ax + By = C\) and state values for \(a, b, c\).
Experiential Activity One Answers

1. \( y = -4x + 7 \)

2. \( y = -3x + \frac{7}{2} \)

3. \( 5x + y = -3 \)

4. \( 2x - 3y = 1 \)

5. \( 2x - 3y = -17 \)

6. \( y = -\frac{1}{2}x + \frac{1}{4} \)

7. \( 2x - y = 5 \)

8. \( A = 2 \)
   \( B = -3 \)
   \( C = -15 \)

   \( m = \frac{7}{4} \)

9. \( b = \frac{1}{2} \)

10. \( A = 5 \)
    \( B = -4 \)
    \( C = -8 \)
OBJECTIVE TWO

When you complete this objective you will be able to…

Determine the equation of a line using the point-slope form of the line, given two points on the line or given one point and the slope of the line.

Exploration Activity

Point - Slope Equation

To get the Point – Slope equation, we start with the slope formula:

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

The slope formula uses two specific points on a line. If we replace \((x_2, y_2)\) with the general point \((x, y)\) we get:

\[ m = \frac{y - y_1}{x - x_1} \]

which simplifies to:

\[ y - y_1 = m(x - x_1) \] \( \leftrightarrow \) Equation (1) This is the Point Slope Equation !!

Equation (1) is the point-slope form of the equation of a line and is very important in any study of linear equations. What it means is that given any two points on the line or given one point and the slope of the line we can solve for the equation of the line.

EXAMPLE 1

Find the standard equation of the line that passes through the point \((6, -20)\) and that has a slope of \(\frac{1}{4}\).

SOLUTION:

Since \(m = \frac{1}{4}\) and \((x_1, y_1) = (6, -20)\) we can substitute directly into equation (1) above:

\[ y - (-20) = \frac{1}{4}(x - 6) \]

\[ y + 20 = \frac{1}{4}(x - 6) \]

multiply both sides by 4

\[ 4y + 80 = x - 6 \]

\[ -x + 4y = -86 \]

we generally make the \(x\) coefficient positive, so

\[ x - 4y = 86 \]

Thus \(x - 4y = 86\) is the equation of the line.
EXAMPLE 2

Determine the equation of the line passing through the points \((-1, 6)\) and \((8, 40)\).

SOLUTION:

First we must find the slope

\[
m = \frac{6 - 40}{-1 - 8} = \frac{34}{9}
\]

and then taking either point as \((x_1, y_1)\) and the slope, we substitute into (1), and get:

\[
y - 40 = \frac{34}{9} (x - 8)
\]

\[
9y - 360 = 34x - 272
\]

\[
-34x + 9y = 88
\]

\[
34x - 9y = -88 \quad \leftrightarrow \text{the equation of the line.}
\]

NOTE:

The student should realize that being given 2 points on the line is essentially the same thing as being given one point and the slope.
Experiential Activity Two

Determine the equations of the following straight lines:

1. Passing through (4, −6) and (10, 10). Show Me.
2. Passing through (−100, 0) and (−50, 8).
3. Passing through (5, 30), and (11, −40).
4. Passing through (6, −15) with slope equal to $\frac{1}{3}$.
5. Passing through (−8, 20) with slope equal to −6.
6. Passing through (80, −1) with slope equal to $-\frac{1}{2}$.

Experiential Activity Two Answers

1. $8x − 3y = 50$
2. $4x − 25y = −400$
3. $35x + 3y = 265$
4. $x − 3y = 51$
5. $6x + y = −28$
6. $x + 2y = 78$
OBJECTIVE THREE

When you complete this objective you will be able to...

Determine the slope of an oblique line given its equation.

Exploration Activity

EXAMPLE 1

Determine slope of the line given represented by the equation $3x - 2y = 1$

SOLUTION:

METHOD 1

Find any two points on the line and then apply the slope formula.

Take $x = 2$ then $3(2) - 2y = 1$

$6 - 2y = 1$

$-2y = -5$

$y = \frac{5}{2}$

So $\left(2, \frac{5}{2}\right)$ is a point, let's call it $(x_1, y_1)$

Now take $x = 0$ then $3(0) - 2y = 1$

$-2y = 1$

$y = -\frac{1}{2}$

So $\left(0, -\frac{1}{2}\right)$ is another point, let's call it $(x_2, y_2)$
Now apply the slope formula

\[ m = \frac{y - y_1}{x - x_1} \]

\[ m = \frac{1 - 5}{2 - 2} \]

\[ m = \frac{3}{2} \]

the slope is \( \frac{3}{2} \) which means the line rises 3 units for every 2 units it moves horizontally.

**METHOD 2**

Use the \( y \)-intercept form of the equation of the line to determine the slope of a line.

Change the given equation \( 3x - 2y = 1 \) into the form \( y = mx + b \) where the coefficient of \( x \) represents the slope of the line.

To do this solve for \( y \):

\[ 3x - 2y = 1 \]

\[ -2y = -3x + 1 \]

\[ y = \frac{-3}{-2} x + \frac{1}{-2} = \frac{3}{2} x - \frac{1}{2} \]

Now compare \( y = \frac{3}{2} x - \frac{1}{2} \) with \( y = mx + b \)

and see here that:

\[ m = \frac{3}{2} \] is the slope, and

\[ b = -\frac{1}{2} \] is the \( y \)-intercept.

**Note:** For this method to work the coefficient of \( y \) must always be +1.

For slope questions obtained from the computer based question bank always reduce the fraction to lowest terms before identifying rise and run. Also put rise as + or −, and put run as +.

**COMMENT:**

Of the two methods just presented, Method 2 is by far the more efficient way to go.
Experiential Activity Three

1. Determine the slope of the line described by $5x - 8y = 32$. Show Me.

2. Determine the slope of the line described by $6x + 5y = -25$.

3. Determine the slope of the line described by $2y = 3x + 6$.

4. Determine the slope of the line described by $52x - 11y = 1$

5. Determine the slope of the line described by $x + 21y = 80$.

Experiential Activity Three Answers

1. $\frac{5}{8}$

2. $\frac{-6}{5}$

3. $\frac{3}{2}$

4. $\frac{52}{11}$

5. $\frac{-1}{21}$
OBJECTIVE FOUR

When you complete this objective you will be able to…

Determine the equation of a line given its graph.

Exploration Activity

EXAMPLE 1

Given:

FIND: The slope of the given line.

METHOD 1

SOLUTION:

Given \( p_1 \) is (1, 0) and \( p_2 \) is (5, 2), then we use the slope formula to get:

\[
\text{slope of } p_1p_2 = m = \frac{2 - 0}{5 - 1} = \frac{1}{2}
\]
Now let us pick a general point \((x, y)\) on the line and substitute it into the slope formula for \((x_2, y_2)\) to obtain:

\[
m = \frac{1}{2} = \frac{y - 2}{x - 5}
\]

\[1(x - 5) = 2(y - 2)\]

\[x - 5 = 2y - 4\]

\[x - 2y = 1 \quad \text{the desired equation in Standard form.}\]

**METHOD 2**

Apply the formula:

\[y - y_1 = m(x - x_1)\]

**Given:**

\[p_1 = (1, 0) \quad \text{here } x_1 = 1 \text{ and } y_1 = 0\]

\[p_2 = (5, 2) \quad \text{here } x_2 = 5 \text{ and } y_2 = 2\]

To find the slope we use the formula \(m = \frac{y_2 - y_1}{x_2 - x_1}\)

so the slope \(p_1 p_2 = \frac{1}{2}\)

\[
\begin{align*}
&= \frac{2 - 0}{5 - 1} \\
&= \frac{1}{2}
\end{align*}
\]

Substitute \(m = \frac{1}{2}\) and \((x_1, y_1) = (1, 0)\) into the formula \(y - y_1 = m(x - x_1)\):

\[y - 0 = \frac{1}{2}(x - 1)\]

\[2y = x - 1\]

\[-x + 2y = -1\]

\[x - 2y = 1 \quad \text{Standard form.}\]

**NOTE:** This method is the more commonly used method.
Experiential Activity Four
Express answers in the form indicated.

1. Determine the equation of \( l_1 \) in standard form. Show Me.

2. Determine the equation of \( l_2 \) in standard form.

3. Determine the equation of \( l_3 \) in the form \( y = mx + b \).

Experiential Activity Four Answers

1. \( 5x + 6y = -30 \)

2. \( x - y = 0 \)

3. \( y = -\frac{1}{2}x \)
**OBJECTIVE FIVE**

*When you complete this objective you will be able to…*

Determine the slope, $x$ intercept, $y$ intercept, and/or the equation of a horizontal or vertical line.

**Exploration Activity**

**HORIZONTAL LINE ($l_1$)**

By definition any two points on a horizontal line have the same second coordinate, i.e. the same $y$ coordinate.

(on $l_1$ all $y$’s are 3)

The SLOPE of a horizontal line ALWAYS = 0.

**NOTE:** $l_1$ and $l_2$ refer to the previous figure.

- $x$ intercept of $l_1$  
  $l_1$ does not cross $x$ - axis so it does not have an $x$ intercept

- $y$ intercept of $l_1$  
  $l_1$ crosses $y$ axis at $y = 3$ so $y$ intercept is 3.

- **Equation of $l_1$**  
  All points on $l_1$ have a $y$ value of 3 so the equation of $l_1$ is $y = 3$. ($x$ can be any real number).
VERTICAL LINE($l_2$)

By definition any two points on a vertical line have the same first coordinates (on $l_2$ all $x$'s are $-2$).

Slope of $l_2$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{3 - (-5)}{-2 - (-2)}$$

$$= \frac{8}{0}$$

= undefined

The slope of a vertical line is always undefined.

$x$ intercept of $l_2$

$l_2$ crosses the $x$ axis at $x = -2$ so the $x$ intercept is $-2$.

$y$ intercept of $l_2$

$l_2$ does not cross the $y$ axis so it does not have a $y$ intercept.

Equation of $l_2$

All points on $l_2$ have an $x$-value of $-2$ so the equation of $l_2$ is $x = -2$. ($y$ can be any real number).

**Important:** Summary of Objective 5

1. Every horizontal line has a slope of zero.
2. Every vertical line has an undefined slope.
3. Horizontal lines do not have $x$ intercepts.
4. Vertical lines do not have $y$ intercepts.
5. Every horizontal line has an equation of the form $y = k$, where $k$ is the $y$ intercept. For example, $y = 3$ is the equation of a horizontal line which crosses the $y$ axis at 3.
6. Every vertical line has an equation of the form $x = k$, where $k$ is the $x$ intercept. For example, $x = -6$ is the equation of a vertical line which crosses the $x$ axis at $-6$. 
Experiential Activity Five

From the previous figure:

1. Determine the following for \( l_1 \)
   a. slope
   b. \( x \) intercept
   c. \( y \) intercept
   d. equation in standard form

2. Determine the following for \( l_2 \)
   a. slope
   b. \( x \) intercept
   c. \( y \) intercept
   d. equation in standard form

3. Determine the equation of the line passing through the points (6, -7) and (6,14).
   \[ \text{Show Me.} \]

4. Determine the equation of the line passing through the points (3, -2) and (-4, -2).
   \[ \text{Show Me.} \]

5. Determine the equation of the line passing through the points (-14, 0) and (-14,11).
Experiential Activity Five Answers

1.   a) 0  
     b) none  
     c) 2  
     d) $y = 2$

2.   a) undefined  
     b) $-1$  
     c) none  
     d) $x = -1$

3.   $x = 6$

4.   $y = -2$

5.   $x = -14$
OBJECTIVE SIX

*When you complete this objective you will be able to…*

Apply the relationships that occur between lines that are parallel and those that occur between lines that are perpendicular.

**Exploration Activity**

**Parallel lines:**
Parallel lines are two lines that extend in the same direction but do not intersect. Parallel lines have the same slope.

**Perpendicular lines:** Perpendicular lines are two lines that intersect at 90°. The slopes of perpendicular lines are the negative reciprocals of each other. (product of the two slopes $= -1$)

**EXAMPLE 1**

Given the slope of $l_1 = \frac{2}{3}$, and $l_1$ is parallel to $l_2$, and $l_2$ is perpendicular to $l_3$.

Find the slopes of $l_2$ and $l_3$

**SOLUTION:**

1. Slope $l_1 = $ slope $l_2 \iff$ parallel lines, slopes are equal.

   Therefore slope $l_2 = \frac{2}{3}$

2. Slope $l_2 = $ negative reciprocal $l_3$ (perpendicular lines)

   Therefore slope $l_3 = -\frac{3}{2}$
EXAMPLE 2

Given line $l_1$ with equation $5x + 2y = -7$, find the slopes of $l_2$ and $l_3$ if $l_2$ is parallel to $l_1$ and $l_3$ is perpendicular to $l_1$.

SOLUTION:

$$5x + 2y = -7$$

Change the equation to the form $y = mx + b$ so we can identify the slope, $m$.

$$2y = -5x - 7$$

$$y = -\frac{5}{2}x - \frac{7}{2}$$

Since the coefficient of $x$ is the slope we have:

Slope $l_1 = -\frac{5}{2}$

Slope $l_2 = -\frac{5}{2}$ ($l_1$ is parallel to $l_2$)

Slope $l_3 = \frac{2}{5}$ ($l_1$ is perpendicular to $l_3$)
Experiential Activity Six

1. Determine the equation, in the form $Ax + By = C$, of the line through $(8, 7)$ and parallel to the $x$-axis.

2. Determine the equation, in the form $Ax + By = C$, of the line through $(-3,2)$ and perpendicular to the $y$-axis.

3. Determine the equation, in the form $y = mx + b$, of the line through the origin and parallel to the line $x + 2y + 3 = 0$.

4. Determine the equation, in the form $Ax + By = C$, of the line having the same $x$ intercept as the line $5x - 2y = 5$ and parallel to $2x - 3y = -5$.

5. Determine the equation, in the form $Ax + By = C$, of the line containing $(-4, 3)$ and parallel to $y = 1$.

6. Determine the equation, in the form $y = mx + b$, of the line through $(-2, 5)$ and perpendicular to $2x + 5y - 2 = 0$. Show Me.

7. Determine the equation, in the form $Ax + By = C$, of the line having the same $y$ intercept as the line $3x - 2y = 5$ and parallel to $8x - 5y = 3$.

Experiential Activity Six Answers

1. $y = 7$

2. $y = 2$

3. $y = -\frac{1}{2}x$

4. $2x - 3y = 2$

5. $y = 3$

6. $y = \frac{5}{2}x + 10$

7. $16x - 10y = 25$
OBJECTIVE SEVEN

When you complete this objective you will be able to…

Solve word problems relating to straight lines.

Exploration Activity

EXAMPLE 1

1. Determine the perimeter of a triangle with vertices \( A(1, 4) \), \( B(5, 1) \), \( C(1, -2) \).

SOLUTION:

Find the distance \( AB \) using the distance formula:

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\( x_1 = 1, y_1 = 4, x_2 = 5, y_2 = 1 \)

\[
d(A, B) = \sqrt{(5 - 1)^2 + (1 - 4)^2} = 5
\]
Find the distance $BC$ using the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$x_1 = 5$, $y_1 = 1$, $x_2 = 1$, $y_2 = -2$

$$d(B, C) = \sqrt{(1 - 5)^2 + (-2 - 1)^2} = 5$$

Find distance $AC$, it's a vertical line therefore the distance $= y_2 - y_1$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$x_1 = 1$, $y_1 = -4$, $x_2 = 1$, $y_2 = -2$

$$d(A, C) = \sqrt{(1 - 1)^2 + (-2 - 4)^2} = 6$$

Therefore the perimeter of triangle $ABC$

$$= d(A, B) + d(B, C) + d(A, C)$$

$$= 5 + 5 + 6$$

$$= 16 \text{ units}$$
Experiential Activity Seven

1. A triangle has vertices $A(-6, 5)$, $B(4, -1)$ and $C(2, 3)$. Determine the length of the median from $C$. (A median joins a vertex to the midpoint of the opposite side).

2. A line segment has a midpoint at $(2, -3)$ and one endpoint at $(5, -8)$. Determine the other endpoint of the line.

3. The line through $A(x, -2)$ and $B(-3, 4)$ is parallel to the line through $C(-1, 4)$ and $D(1, -2)$. Determine the value of $x$. Show Me.

4. The points $T(-1, 2)$, $Q(4, -3)$, and $R(-2, y)$ are on the same line (they are collinear). Determine the value of $y$.

Experiential Activity Seven Answers

1. $\sqrt{10}$

2. $(-1, 2)$

3. $x = -1$

4. $y = 3$
Practical Application Activity

Complete the Linear Equation - 2 module assignment in TLM.

Summary

The topics presented in this Module dealt primarily with the algebra of equations of straight lines.

By that we mean we studied:

1. Given the information about the line: find its equation.
2. Given the equation of a line: find information about it.

Straight-line theory flows over into many varieties of applied problems.